Computing moment polytopes of tensors

with applications in algebraic complexity and quantum information

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together with

Matthias Christandl, Vladimir Lysikov, Harold Nieuwboer, Michael Walter and Jeroen Zuiddam

Introduction

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central are **relations**, such as $S = (A \otimes B \otimes C)\, T$ (restriction), $A \in \mathbb{C}^{a\times a}$, $B \in \mathbb{C}^{b\times b}$, $C \in \mathbb{C}^{c\times c}$

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The **moment polytope** of a tensor contains such information,
both geometric and asymptotic representation-theoretic

Introduction: relevance

The **moment polytope** of a tensor contains geometric & representation theoretic information

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- Strassen's duality theory for asymptotic rank, to construct the **quantum functionals** ↔ asymptotic rank conjecture
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- non-commutative optimization through **scaling algorithms**, which optimize over such polytopes
  [Bürgisser-Franks-Garg-Oliveira-Walter-Wigderson, FOCS 2019]

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Obtaining **new examples** is <u>an important step</u> towards new **theoretic results**

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- **separations** between moment polytopes of matrix multiplication and unit tensors (of any sizes)
- a **no-go result** for an operational property of moment polytopes relating to asymptotic restriction
- the first-ever **constructions** of non-free tensors

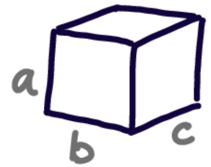
Outline

- What are moment polytopes?
- How can we compute them?
- Applications

What are moment polytopes?

Moment polytopes: first definition

Fix a **tensor** $T \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$



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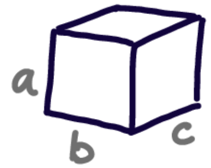

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- flatten to a **matrix** $T_1 \in \mathbb{C}^a \otimes (\mathbb{C}^b \otimes \mathbb{C}^c)$

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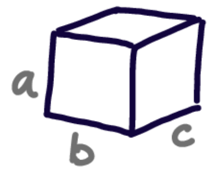

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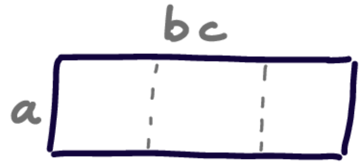
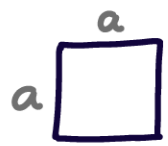


- consider $\quad T_1 T_1^* \in \mathbb{C}^a \otimes \mathbb{C}^a$



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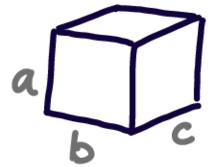

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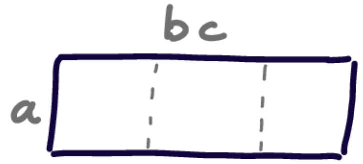

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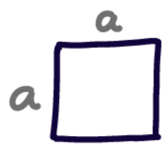

- consider $T_1 T_1^* \in \mathbb{C}^a \otimes \mathbb{C}^a$

- and look at the **normalized eigenvalues** $r_1(T) = (\alpha_1 \overset{\geq}{,} \alpha_2 \overset{\geq}{,} \dots \overset{\geq}{,} \alpha_a)$

↳ $\sum_i \alpha_i = 1$

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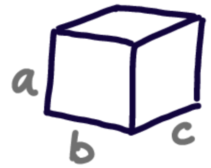

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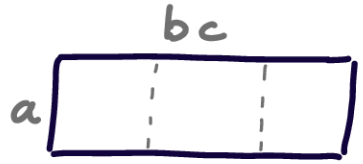

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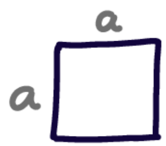

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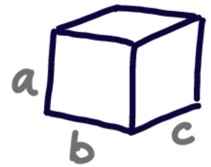

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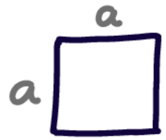

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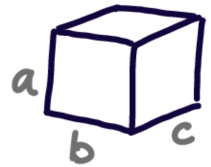

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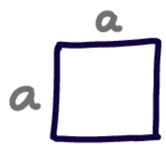

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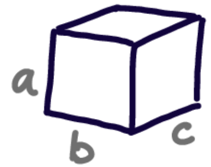

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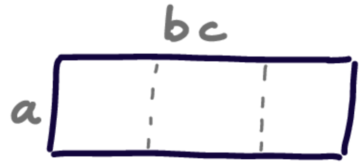

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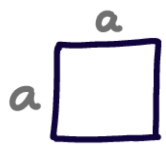

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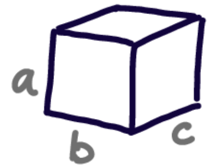

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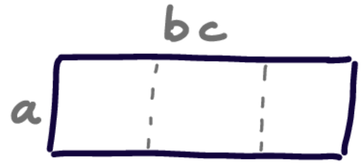

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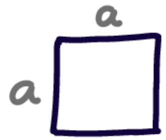

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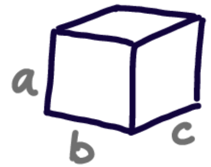

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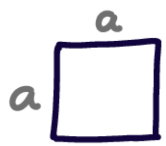

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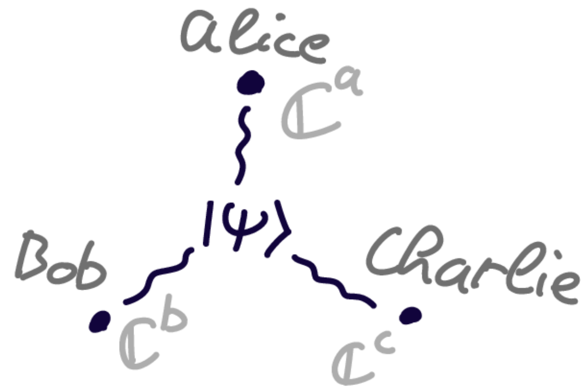
is a **multipartite quantum state**



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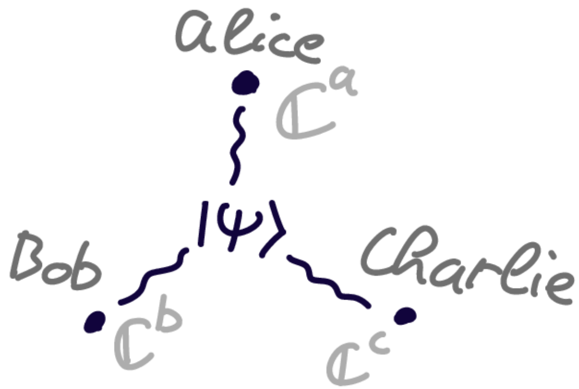

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stochastic local operations
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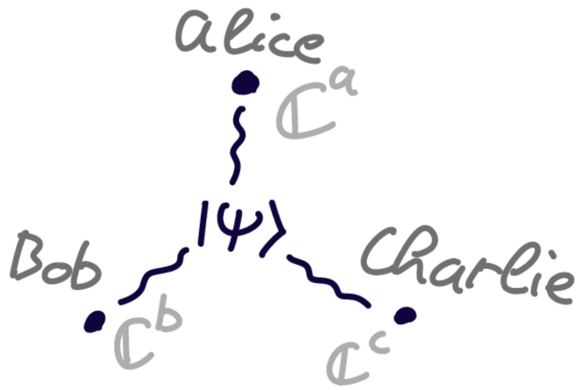

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- Set $\underset{T' = |\psi\rangle}{\rho = |\psi\rangle\langle\psi|}$, then $r_1(T') = \mathrm{spec}(\rho_{Alice})$

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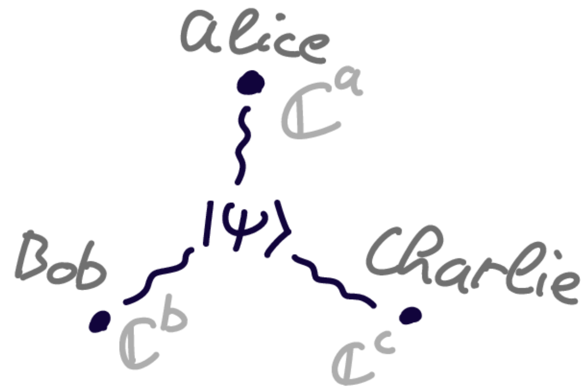
$$|\varphi\rangle := \frac{T}{\|T\|} \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$$



is a **multipartite quantum state**

- $A \otimes B \otimes C$ is a SLOCC operation

  stochastic local operations & classical communication

  "operations that do not increase entanglement"

- Set $\rho = |\psi\rangle\langle\psi|$, then $r_1(T') = \mathrm{spec}(\rho_{\text{Alice}})$

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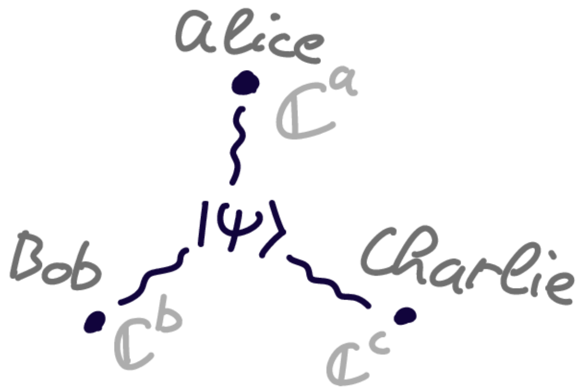

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$\Longrightarrow$ it is called the **entanglement polytope** of $|\psi\rangle$

Theorem [Mumford-Ness 1984]

$\Delta(T)$ is a rational (convex compact) polytope

based on work in
- symplectic geometry
- invariant theory
- representation theory

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Examples

$$\Delta\Big(\sum_{i=1}^{2} e_i\otimes e_i\otimes e_i\Big) = \mathrm{conv}\left\{\begin{array}{l}
(\overbrace{1,\,0}^{\mathbb{R}^2},\ \overbrace{1,\,0}^{\mathbb{R}^2},\ \overbrace{1,\,0}^{\mathbb{R}^2}) \\
(\tfrac12,\tfrac12,\ \tfrac12,\tfrac12,\ \tfrac12,\tfrac12) \\
(1,0,\ \tfrac12,\tfrac12,\ \tfrac12,\tfrac12) \\
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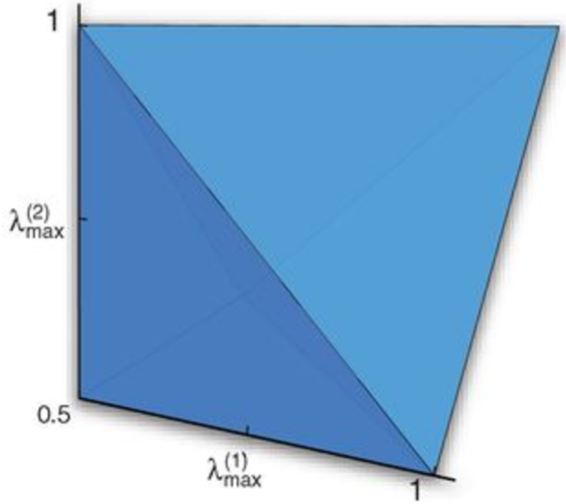
(\tfrac12,\tfrac12,\ \tfrac12,\tfrac12,\ 1,0)
\end{array}\right\}$$

↳ unit tensor (order 2)



$$\Delta\left(\begin{array}{l} e_1\otimes e_1\otimes e_2 \\ +e_1\otimes e_2\otimes e_1 \\ +e_2\otimes e_1\otimes e_1\end{array}\right) = \mathrm{conv}\left\{\begin{array}{l}
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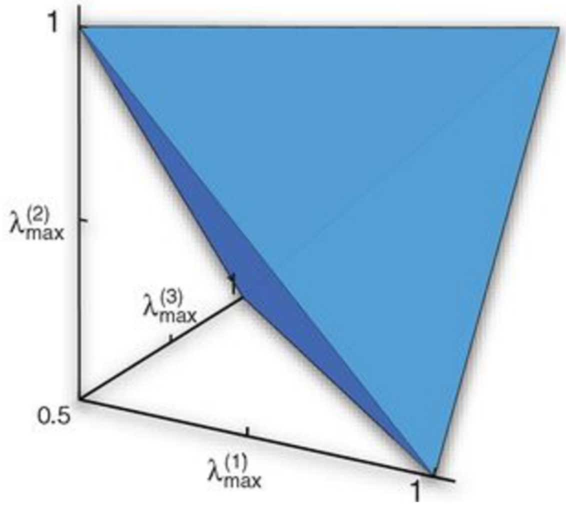
↳ W tensor

slices: $\begin{matrix}0&1\\1&0\end{matrix}\quad\begin{matrix}1&0\\0&0\end{matrix}$



Computing moment polytopes

Franz' description

Franz' description

Def. (support)

$$\operatorname{supp}(T) := \{ (e_i, e_j, e_k) \mid T_{i,j,k} \neq 0 \}$$

$$\subseteq \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c$$

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Example:

$$\mathrm{supp}(W) = \mathrm{supp}\begin{pmatrix} e_1 \otimes e_1 \otimes e_2 \\ + e_1 \otimes e_2 \otimes e_1 \\ + e_2 \otimes e_1 \otimes e_1 \end{pmatrix} = \left\{ \begin{matrix} (1,0\,,\,1,0\,,\,0,1), \\ (1,0\,,\,0,1\,,\,1,0), \\ (0,1\,,\,1,0\,,\,1,0) \end{matrix} \right\}$$

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i.e. we can take it randomly

an algorithm

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dutch national supercomputer

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... and didn't terminate

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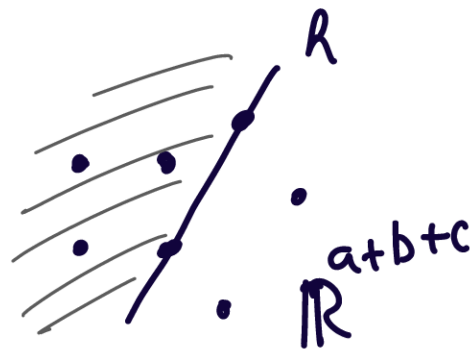

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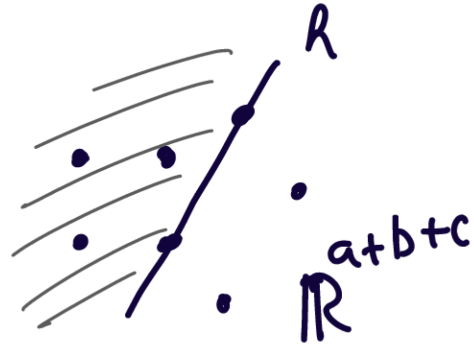

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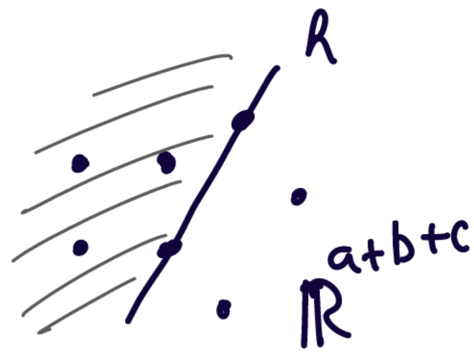

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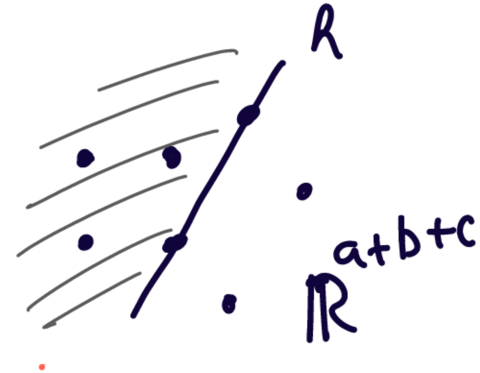

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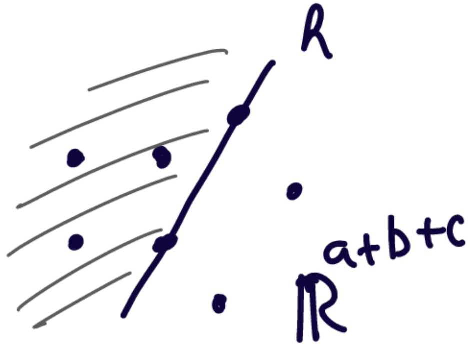

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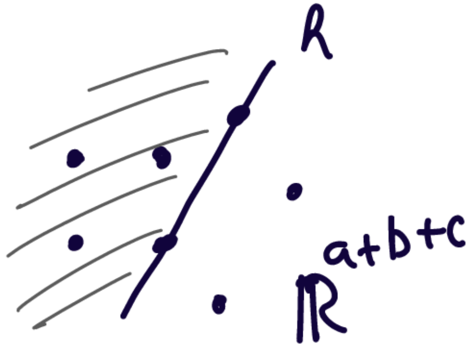

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We can do tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ in seconds !!

What about $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$?

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Many Optimizations

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e.g. using

Many Optimizations

→ symmetries of supports

→ Gröbner over prime fields

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→ using known inequalities for maximal polytopes

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↓

We can do it

Applications

Results : $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

Theorem

We know all moment polytopes in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

↳ there are 29 different ones

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They satisfy

- $F_\theta(T \oplus S) = F_\theta(T) + F_\theta(S)$
- $F_\theta(T \otimes S) = F_\theta(T)\, F_\theta(S)$
- $F_\theta\big((A \otimes B \otimes C)\, T\big) \leq F_\theta(T)$
- $F_\theta(e_1 \otimes e_1 \otimes e_1) = 1$

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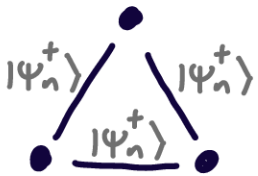
$MM_n =$ 

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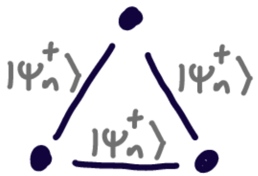
$MM_n =$ 

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- The proof relates MM_n with **polynomial multiplication tensors**

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$\hookrightarrow$ so quantum functionals "pick out special information"

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This tensor in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ is not free:

$$\left[\begin{array}{ccc|ccc|ccc} & & & & & 1 & 1 & & \\ & & 1 & & 1 & & & & \\ 1 & 1 & & & & & & & \end{array}\right], \quad \left[\begin{array}{cccc|cccc|cccc|cccc} & & & & & & & & & & & 1 & 1 & & & \\ & & & & & & & 1 & & 1 & & & & & & \\ & & & 1 & & & 1 & & & & & & & & & \\ 1 & 1 & 1 & & & & & & & & & & & & & \end{array}\right], \quad \text{etc.}$$

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Thanks

Moment polytopes: representation theory

The group $GL_a \times GL_b \times GL_c$ acts on $(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)^{\otimes n}$

$$(A, B, C) \longmapsto (A \otimes B \otimes C)^{\otimes n}$$

Rep. theory tells us:

$$(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)^{\otimes n} = \bigoplus_{\lambda,\mu,\nu} V_{\lambda,\mu,\nu}$$

- $V_{\lambda,\mu,\nu}$ are isotypic subspaces
- labels are **partitions**: $\lambda \in \mathbb{N}^a$ $\quad \sum_i \lambda_i = n$ $\quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_a$

projector onto

Def.

$$\Delta^{\mathrm{repr}}(T) := \overline{\left\{ \left(\frac{\lambda}{n}, \frac{\mu}{n}, \frac{\nu}{n} \right) \;\middle|\; P_{\lambda,\mu,\nu}\, T^{\otimes n} \neq 0 \,,\; n > 0 \right\}}$$

$$\subseteq \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c$$

Computing entanglement polytopes is difficult

- Only known completely for $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 (\otimes \mathbb{C}^2)$
  + some sporadic examples

Aside: a decision problem

> **Problem:** Given $T \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ & $p \in \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c$, determine whether $p \in \Delta(T)$

- "Close" to (N)P & coNP
  - ↳ bitsize of certificates is a problem
- Scaling methods can decide yes-instances in practice
- # vertices is typically not polynomial in dimension (a,b,c)

- Better understanding requires <u>more examples</u>

Some stats

$\in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

	Inequalities					Vertices	Runtimes	
Tensor	All	Not generic	Maxranks	Attainable	Final	Final	$\mathbb{Q}$	$\mathbb{F}_q$
$\langle 3 \rangle$	2845	2187	355	0	45	33	0.254	0.239
T_4	2845	2187	355	20	52	53	0.264	0.237
T_9	2845	2187	736	292	25	18	0.293	0.263
$\langle 4 \rangle$	8109383	7139405	1102518	0	270	328	-	20:10
$\mathrm{MM}_{2,2,2}$	8109383	7139405	1102518	1227	129	181	-	3:06

$\in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$

this step uses Gröbner bases

still going after 10 hours